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GENERALIZING A THEOREM OF P. HALL ON FINITE-BY-NILPOTENT GROUPS

GUSTAVO A. FERNÁNDEZ-ALCOBER AND MARTA MORIGI

Abstract. Let $\gamma_i(G)$ and $Z_i(G)$ denote the i-th terms of the lower and upper central series of a group G, respectively. In [2] P. Hall showed that if $\gamma_{i+1}(G)$ is finite then the index $|G : Z_{2i}(G)|$ is finite. We prove that the same result holds under the weaker hypothesis that $|\gamma_{i+1}(G) : \gamma_{i+1}(G) \cap Z_i(G)|$ is finite.

1. Introduction

If G is an arbitrary group, a classical theorem of Schur asserts that if the center $Z(G)$ has finite index in G then derived subgroup G' is finite. This was later generalized by Baer (see 14.5.1 in [5]) to any term of the lower central series; namely, if $|G : Z_i(G)|$ is finite then $\gamma_{i+1}(G)$ is finite. The converse does not hold in general, anyway in [2] P. Hall proved that if $\gamma_{i+1}(G)$ is finite then $|G : Z_{2i}(G)|$ is finite. For the case $i = 1$ a stronger result is known to hold: actually, if $|G' : G' \cap Z(G)|$ is finite then $|G : Z_2(G)|$ is also finite. This result was obtained independently by the first author and Moretó (see Theorem E of [1]) and by Podoski and Szegedy in [4]. In this paper we show that this last property can be extended to an arbitrary value of i. More precisely, the following is true.

Theorem A. *Let G be a group such that $|\gamma_{i+1}(G) : \gamma_{i+1}(G) \cap Z_i(G)|$ is finite. Then $|G : Z_{2i}(G)|$ is finite.*

From the proof of the theorem it can be checked that $|G : Z_{2i}(G)|$ is bounded in terms of $|\gamma_{i+1}(G) : \gamma_{i+1}(G) \cap Z_i(G)|$, the ultimate reason being that Lemma 2.1 and Theorem 2.2 below can be stated in a quantitative version. However, we have made no attempt at giving a sharp bound. We will mention here that in the case $i = 1$, the existence of such a bound was proved by Isaacs in [3] when the group G is finite, and then an explicit bound was given in [4] for an arbitrary group G.

Related to Theorem A, the following two questions arise naturally:

(1) To what extent is Theorem A best possible? If the weaker condition that $|\gamma_{i+1}(G) : \gamma_{i+1}(G) \cap Z_{i+1}(G)|$ is finite holds, can we conclude that $|G : Z_{2i}(G)|$ is finite?
(2) In the case that $\gamma_{i+1}(G)$ is finite, if G is also finitely generated, then a stronger result holds, namely $|G : Z_i(G)|$ is finite. Is this true also under

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the hypothesis of Theorem A? Does it follow at least that $|G:Z_j(G)|$ is finite for some j smaller than $2i$?

The answer to both these questions is negative. To see this, for arbitrary c, consider a finitely generated nilpotent group G of class c in which the upper and lower central series coincide and such that $|G:Z_{c-1}(G)|$ is infinite. For example, one can take the semidirect product $G = B \ltimes A$, where $B = \langle b \rangle$ is an infinite cyclic group, A is the free abelian group on free generators $a_1, \dots, a_c$, and b acts on A by $a_i^b = a_i a_{i+1}$ for $1 \le i \le c-1$ and $a_c^b = a_c$. Now if $i \ge 1$ is any fixed integer, we get counterexamples to the first and the second questions by choosing $c = 2i+1$ and $c = 2i$, respectively.

Finally, we observe that combining our result with Baer's theorem it follows that if $|\gamma_{i+1}(G) : \gamma_{i+1}(G) \cap Z_i(G)|$ is finite then $\gamma_{2i+1}(G)$ is also finite. Actually, one of the key arguments in our proof of Theorem A is the following generalization of this fact, which might be interesting in its own right.

Theorem B. *Let G be a group such that $|\gamma_s(G) : \gamma_s(G) \cap Z_t(G)|$ is finite for some s, t. Then $|\gamma_{s+j}(G) : \gamma_{s+j}(G) \cap Z_{t-j}(G)|$ is finite for every j such that $0 \le j \le t$. In particular, $\gamma_{s+t}(G)$ is finite.*

2. The results

The notation we use is standard. Moreover, following the book [5], if A and B are subgroups of a group G and n is a natural number, we define recursively:

$$[A, {}_1B] = [A, B], \qquad [A, {}_nB] = [A, {}_{n-1}B, B].$$

Throughout the paper, we will repeatedly use the following well-known result (see for instance 5.1.10 in [5]).

Three Subgroup Lemma. *Let H, K, L be subgroups of a group G. If two of the commutator subgroups $[H, K, L], [K, L, H], [L, H, K]$ are contained in a normal subgroup of G, then so is the third.*

Another result which will be often used in our proofs is stated for convenience in the following lemma, whose proof is elementary. Most of the times, we will apply it modulo a normal subgroup.

Lemma 2.1. *Let H, K be subgroups of a group G. If $[H, K]$ is finite and H is finitely generated, then the centralizer $C_K(H)$ has finite index in K.*

We will also need the following result of Baer (see for instance 14.5.2 in [5]).

Theorem 2.2. *Let $M \le H$ and $N \le K$ be normal subgroups of a group G such that $|H : M|$ and $|K : N|$ are finite, and $[H, N] = 1 = [K, M]$. Then $[H, K]$ is finite.*

The key step in the proof of our main theorem is in the following proposition.

Proposition 2.3. *Let G be a group and let $s \ge 1$ be an integer such that $|\gamma_s(G) : \gamma_s(G) \cap Z(G)|$ is finite. Then $C_G(\gamma_s(G))$ has finite index in G and $\gamma_{s+1}(G)$ is finite.*

Proof. Let $Z = \gamma_s(G) \cap Z(G)$. As $\gamma_s(G)/Z$ is finite, there exists a finitely generated subgroup U of G such that $\gamma_s(G) = \gamma_s(U)Z$. By applying P. Hall's theorem to the quotient group $G/Z(G)$, we obtain that $|G : Z_{2s-1}(G)|$ is finite. By the theorem of Baer mentioned in the introduction, it follows that $\gamma_{2s}(G)$ is finite. Since

$\gamma_k(U)/\gamma_{k+1}(U)$ is finitely generated for every $k = 1, \dots, 2s-1$, we conclude that all terms of the lower central series of U are finitely generated.

We are going to prove that, for every $j = 1, \dots, s$, there exists a subgroup H_j of finite index in $\gamma_j(G)$ such that $[H_j, \gamma_{s-j+1}(U)] = 1$. Then $[H_1, \gamma_s(G)] = [H_1, \gamma_s(U)Z] = [H_1, \gamma_s(U)] = 1$, which proves that $|G : C_G(\gamma_s(G))|$ is finite.

We prove the existence of H_j by reverse induction on j. For $j = s$, we take $H_s = Z$. Suppose now that we already have H_{j+1} of finite index in $\gamma_{j+1}(G)$ such that $[H_{j+1}, \gamma_{s-j}(U)] = 1$, and let us see how to construct the subgroup H_j. Let $K_j = C_{\gamma_j(G)}(\gamma_{s-j}(U)Z/Z)$. Since $\gamma_{s-j}(U)Z/Z$ is finitely generated and $[\gamma_j(G), \gamma_{s-j}(U)]Z/Z \le \gamma_s(G)/Z$ is finite, it follows from Lemma 2.1 that K_j has finite index in $\gamma_j(G)$. Also

$$[K_j, \gamma_{s-j}(U), U] \le [Z, U] = 1. \tag{1}$$

Let now $D_{j+1} = C_{\gamma_{j+1}(G)}(\gamma_{s-j}(U))$. Since $[H_{j+1}, \gamma_{s-j}(U)] = 1$, we have $H_{j+1} \le D_{j+1}$ and consequently $|\gamma_{j+1}(G) : D_{j+1}|$ is finite. Consider $T_j = \gamma_j(G)U$. We claim that D_{j+1} is normal in T_j. On the one hand, for every u in U we have $[D_{j+1}^u, \gamma_{s-j}(U)] = [D_{j+1}, \gamma_{s-j}(U)]^u = 1$, so that U normalizes D_{j+1}. On the other hand, we have $[\gamma_j(G), \gamma_{s-j}(U), D_{j+1}] \le [\gamma_s(G), D_{j+1}] = [\gamma_s(U)Z, D_{j+1}] = 1$ and $[\gamma_{s-j}(U), D_{j+1}, \gamma_j(G)] = 1$ by the definition of D_{j+1}. By the Three Subgroup Lemma, $[D_{j+1}, \gamma_j(G), \gamma_{s-j}(U)] = 1$ and consequently also $\gamma_j(G)$ normalizes D_{j+1}.

Now we work in the quotient group T_j/D_{j+1}. Since $[\gamma_j(G), U]D_{j+1}/D_{j+1} \le \gamma_{j+1}(G)/D_{j+1}$ is finite and U is finitely generated, it follows that the centralizer L_j of UD_{j+1}/D_{j+1} in $\gamma_j(G)$ has finite index in $\gamma_j(G)$. Observe that

$$[L_j, U, \gamma_{s-j}(U)] \le [D_{j+1}, \gamma_{s-j}(U)] = 1. \tag{2}$$

Finally, let $H_j = K_j \cap L_j$. Then $|\gamma_j(G) : H_j|$ is finite. Moreover, using (1) and (2) and the Three Subgroup Lemma we obtain that $[\gamma_{s-j}(U), U, H_j] = 1$, that is $[\gamma_{s-j+1}(U), H_j] = 1$, as desired.

Now in order to prove that $\gamma_{s+1}(G)$ is finite we apply Theorem 2.2 with $M = C_G(\gamma_s(G))$, $H = G$, $N = \gamma_s(G) \cap Z(G)$ and $K = \gamma_s(G)$. It follows that $[G, \gamma_s(G)] = \gamma_{s+1}(G)$ is finite. □

Let us remark that if N is a normal subgroup of a group G and $|N : N \cap Z(G)|$ is finite, it does not follow that $|G : C_G(N)|$ or $[N, G]$ are finite. For example, let p be a prime and let H and N be two elementary abelian p-groups with countable bases $\{x_i\}_{i \ge 1}$ and $\{y_j\}_{j \ge 0}$, respectively. We define an action of H on N so that x_i centralizes all y_j with $j \ge 1$ and $y_0^{x_i} = y_0 y_i$. Then in the semidirect product $G = H \ltimes N$ we have $|N : N \cap Z(G)| = p$ but both $|G : C_G(N)|$ and $[N, G]$ are infinite.

Corollary 2.4. *Let G be a group such that $|\gamma_s(G) : \gamma_s(G) \cap Z_t(G)|$ is finite for some s, t. Then $|\gamma_{s+j}(G) : \gamma_{s+j}(G) \cap Z_{t-j}(G)|$ is finite for every j such that $0 \le j \le t$. In particular, $\gamma_{s+t}(G)$ is finite.*

Proof. By induction on t it suffices to prove that $|\gamma_{s+1}(G) : \gamma_{s+1}(G) \cap Z_{t-1}(G)|$ is finite. This follows immediately by applying Proposition 2.3 to the quotient group $G/Z_{t-1}(G)$. □

The last part of the proof of Theorem A is inspired from Hall's ideas. The main role will be played by a subgroup C with the two properties that C has finite index in G and $[C, {}_{s-1}G, C] \le Z_{2i-s}(G)$ for every $s \ge 1$, with the convention that

$Z_j(G)=1$ for $j\le 0$. The following technical lemma will ensure that C has the second property.

Lemma 2.5. *Let G be a group and let C_j be the centralizer in G of $\gamma_{i+j}(G)/\gamma_{i+j}(G)\cap Z_{i-j}(G)$ for $j=1,\dots,i$. If $C=\bigcap_{j=1}^i C_j$, then $[C,\,_{s-1}G,C]\le Z_{2i-s}(G)$ for every $s\ge 1$.*

Proof. Observe that C is normal in G and so is $[C,\,_kG]$ for every k. We first prove by induction on k that

$$[[C,\,_kG],\gamma_\ell(G)]\le Z_{2i-k-\ell}(G)\quad\text{for all } k\ge 0 \text{ and for all } \ell\ge i+1. \tag{3}$$

By definition of C, we have $[C,\gamma_\ell(G)]\le Z_{2i-\ell}(G)$ for all $\ell\ge i+1$ and this settles the case $k=0$. Assume now that the statement is true for k. We have $[\gamma_\ell(G),[C,\,_kG],G]\le[Z_{2i-k-\ell}(G),G]\le Z_{2i-k-\ell-1}(G)$. Also, $[G,\gamma_\ell(G),[C,\,_kG]]=[\gamma_{\ell+1}(G),[C,\,_kG]]\le Z_{2i-k-\ell-1}(G)$. So by the Three Subgroup Lemma, it follows that

$$[[C,\,_{k+1}G],\gamma_\ell(G)]=[[C,\,_kG],G,\gamma_\ell(G)]\le Z_{2i-k-\ell-1}(G),$$

which proves the statement for $k+1$.

Now, in order to prove the lemma, we need to show that $[C,\,_{s-1}G,C,\,_{2i-s}G]=1$ for every $s=1,\dots,2i$. We use the formula in the proof of 14.5.4 of [5], which says that if M,N are normal subgroups of a group G then $[[M,N],\,_nG]\le\prod_{\ell=0}^n\big[[M,\,_{n-\ell}G],[N,\,_\ell G]\big]$. Applying this with $M=[C,\,_{s-1}G]$, $N=C$, we have

$$[C,\,_{s-1}G,C,\,_{2i-s}G]\le\prod_{\ell=0}^{2i-s}\big[[C,\,_{2i-\ell-1}G],[C,\,_\ell G]\big].$$

If $0\le\ell\le i-1$ then $2i-\ell\ge i+1$, and since $[C,\,_{2i-\ell-1}G]\le\gamma_{2i-\ell}(G)$, we have $\big[[C,\,_{2i-\ell-1}G],[C,\,_\ell G]\big]=1$ by (3). If $\ell\ge i$ then we can argue similarly, since $[C,\,_\ell G]\le\gamma_{\ell+1}(G)$. $\square$

Theorem 2.6. *Let G be a group such that $|\gamma_{i+1}(G):\gamma_{i+1}(G)\cap Z_i(G)|$ is finite. Then $|G:Z_{2i}(G)|$ is also finite.*

Proof. Let C_j be the centralizer in G of $\gamma_{i+j}(G)/(\gamma_{i+j}(G)\cap Z_{i-j}(G))$ for $j=1,\dots,i$. Since $|\gamma_{i+j}(G):\gamma_{i+j}(G)\cap Z_{i-j+1}(G)|$ is finite by Corollary 2.4, we can apply Proposition 2.3 to the quotient group $G/Z_{i-j}(G)$, and it follows that $|G:C_j|$ is finite. Let $C=\bigcap_{j=1}^i C_j$, which has also finite index in G.

For every $s=1,\dots,i+1$, put $K_s=[C,\,_{s-1}G]$, which is contained in $\gamma_s(G)$. We prove by reverse induction on s that $K_s\cap Z_{2i-s+1}(G)$ has finite index in K_s. For $s=i+1$ the statement is true, as $|K_{i+1}:K_{i+1}\cap Z_i(G)|\le|\gamma_{i+1}(G):\gamma_{i+1}(G)\cap Z_i(G)|$ is finite by hypothesis. Now assume that $Z=K_{s+1}\cap Z_{2i-s}(G)$ has finite index in K_{s+1}. As C has finite index in G, we have $G=\langle g_1,\dots,g_n,C\rangle$ for some $g_1,\dots,g_n\in G$. Let $U=\langle g_1,\dots,g_n\rangle$ and let H_s be the centralizer of UZ/Z in K_s. Since $[K_s,U]Z/Z\le K_{s+1}/Z$ is finite and U is finitely generated, the subgroup H_s has finite index in K_s. Moreover, $[H_s,U]\le Z\le Z_{2i-s}(G)$ and $[H_s,C]\le[K_s,C]\le[C,\,_{s-1}G,C]\le Z_{2i-s}(G)$, where the last inclusion follows from Lemma 2.5. Hence $[H_s,G]=[H_s,UC]\le Z_{2i-s}(G)$ and $H_s\le Z_{2i-s+1}(G)$, which completes the induction.

In particular, for $s=1$ we obtain that $|C:C\cap Z_{2i}(G)|$ is finite. Consequently $|G:Z_{2i}(G)|\le|G:C\cap Z_{2i}(G)|=|G:C|\,|C:C\cap Z_{2i}(G)|$ is finite, and we are done. $\square$

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Matematika Saila, Euskal Herriko Unibertsitatea, 48080 Bilbao, Spain
E-mail address: gustavo.fernandez@ehu.es

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40127 Bologna, Italy
E-mail address: mmorigi@dm.unibo.it